Analytical investigations respecting ***ASTRONOMICAL REFRACTIONS*** *and the application thereof to the formation of convenient* ***TABLES*** *together with the results of observations of circumpolar Stars, tending to illustrate the Theory of Refractions.*

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Read May 9, 1814.

A BRIEF detail will explain the objects of this paper. M. Le Comte Laplace first shewed that the fluxional expression for refraction may be integrated by approximation, as far as about 74° from the zenith, without a knowledge of the variation of density in the atmosphere. *

T. Simpson had deduced by the principles of the 8th section of the first book of Newton's Principia, the fluxional expression for refraction, by considering a particle of light as a body acted on by a force tending to the centre of the earth. † He and others since deduced the integral on the hypothesis, that the density of the atmosphere decreased

* Méc. cél. Liv. 10. c. 1. tom. 4. † Math. Dissertations, p. 51, &c.

uniformly. The simplest form of the integral is that used by Bradley.

Laplace uses the same method of obtaining the fluxional equation as Simpson had done, and then proceeds to investigate the laws of reflection and refraction. He deriveb y an analytical process the conclusions, which Newton had de duced in the 14th section of the first book of the Principia. Laplace next derives his fundamental fluxional expression for refraction which he shews may be integrated as far as 74° from the zenith, without a knowledge of the variation of density in the atmosphere.

In this paper the same fluxional expression, that Laplace obtained, is deduced by a very short method, and by using the common principle of the given ratio of the sines of incidence and refraction. Besides the simplicity of the investigation it has the advantage of avoiding hypothetic principles respecting the rays of light.

The integration of the fluxional expression is also obtained by a method that may be considered as entitled to notice. If the surface of the earth were a plane, then whatever the law of variation of the densities of the different *strata* of air parallel thereto might be, the refraction for any zenith distance would be simply found from the knowledge of the refractive force at the surface, by tbe constant ratio of the sines of incidence and refraction. By the method given this part is separated from the rest, and the effect of the spherical form of the atmosphere is shewn. The formula for refraction

consists of two parts, one the refraction that would take place were the earth a plane, the other the effect due to the spherical form. The latter at 80° zenith distance amounts only to about 12″, and at 40° zenith distance is insensible.

It is shewn that at 80° 45′ the error of the formula deduced cannot amount to half a second, whatever be the variation of density in the atmosphere.

As the approximate formula for refraction as far as about 74° from the zenith is independent of the law of variation of density, it follows that, whatever law be assumed, the same conclusion ought to be deduced as far as about 74°. This is shewn from direct investigation by assuming different laws of variation of density; which beside affording some conclusions useful in our enquiries on this subject, may be considered as interesting.

The results of the experiments of M.M. Biot & Arago on the refractive force of air, and of Mr. Dalton and M. Gai-Lussac on the effects of the change of temperature on the density of air are applied, and a general expression for refraction at any zenith distance less than about 80° obtained, which is entirely independent of astronomical observations.

From this general expression I have formed two tables, by help of which the refraction at any zenith distance less than 80° may be calculated with much convenience.

From a comparison of the co-latitude determined by stars near the pole, and of the same determined by stars more re-

mote, I find, by 525 observations of circumpolar stars, the refraction at 45°, (Bar. 29,60 inches and Therm. 50°.) $= 57'',42$

The same by the French Tables - - $= 57,57$

The same resulting from the direct experiments on the refractive force of air, applied to the formula. - - $= 57,67$

The quantity in the French tables was ascertained from the results of the observations of M. M. Piazzi & Delambre, applied to Laplace's formula by Delambre himself.

My result from the number of observations, from the care used in making them, and from the excellence of my instrument, seems entitled to as much confidence as can be given to a conclusion derived from observations of circumpolar stars, and there is no difference worthy of notice between my result and that of Delambre. But from the nature of the direct experiments on the refractive force of air, the results seem capable of greater exactness than can be derived from observations of circumpolar stars, and therefore strictly perhaps we ought to adopt the result so deduced. However the quantity in the French tables is so nearly equal to this that no inconvenience can arise in the nicest researches in astronomy from adopting these tables.

It is of much importance that the same tables of refraction should be used by astronomers, and it will afford satisfaction to the author of this paper, should it in any manner conduce to this desirable end. It cannot be doubted but that sooner or later the refractions as given by the French tables as far as

80°, or a very slight modification thereof will be generally used by astronomers.

The form of the French tables may not be generally adopted, others more convenient perhaps may be derived. The new form given in this paper will serve as a check in the use of the French tables, and may be thought more convenient than these for observations of the sun, moon and planets.

Below 80° zenith distance, a knowledge of the law of variation of density is absolutely necessary for computing the quantity of refraction. As this cannot be had, all tables for these zenith distances must be in a manner empirical. The French tables are less so than any others, from the method used by Laplace. But the quantity of refraction varies so much from some unexplained cause, the heights of the barometer and thermometer remaining the same, that observations below 80° can be of little use. This irregularity is very manifest at $80^{\circ}\,45'$ in the observed refractions of Capella below the pole. Sixty-five observed refractions of this star are given, and compared with those computed from the formula.

Forty-two observed refractions of α Lyræ below the pole, (zen. dist. $87^{\circ}\,42'$,) are also given. In these the irregularities of refraction are very considerable. The mean of the observed refractions serves for shewing that refraction is greater than would result from a density decreasing uniformly, and less than would result from a uniform tem-

perature. The mean also serves as a criterion of the accuracy of the French and of other tables at this zenith distance.

Investigation of the fluxional equation for refraction.

Let $V R P T$ be the path of a ray of light refracted at P and R, and let CQ be perpendicular to TP produced. (Fig.)

Let the apparent zenith distance HVR	$= \theta$*
CV the radius of the earth	$= a$
CR	$= r'$
CP	$= r$
The density of the air at P	$= \rho$
The density at the surface V	$= (\rho)$
The height of an uniform atmosphere at V	$= l$

Let $m : 1$ represent the ratio of the sine of incidence to the sine of refraction, when light passes from a vacuum into air of the same density as that in VR.

$k' : 1$ the same ratio for air of the density of that in PR, and $k : 1$ the same ratio for air of the density of that in TP

Then it readily appears that

$\sin. VRC : \sin. CRP :: k' : m$

$\sin. CPR : \sin. CPT :: k : k'$

* The same quantities are denoted by the same letters which Laplace has used (chap. 1. liv. 10. tom. 4, Méc. cél.)

Consequently

$a \sin. CVR = r' \sin. VRC = \frac{k'r'}{m} \sin. CRP = \frac{k'r}{m} \sin. RPC$
$= \frac{kr}{m} \sin. OPC.$

Hence $\sin. OPC = \frac{am}{kr} \sin. \theta.$ (1)

This equation is evidently true, whatever be the number of points of refraction between P and V, and therefore is true when VRP is a continued curve as in atmospherical refraction.

The refraction R, that takes place between P and $V =$ the inclination of the lines PT and RV. Hence,

$\dot{R} = \frac{\dot{O}C}{OP}.$

By equation (1) $OC = \frac{am}{k} \sin. \theta.$

The refractive force of air is as its density, and the refractive force in TP is also as k^2-1, (vid. Newton's Optics, book 2, Prop. 10. Horsley's edition, vol. 4, p. 171.)

Therefore let $b\,\rho = k^2-1$, b being a constant quantity Then $k = \sqrt{1+b\rho}$ and $m = \sqrt{1+b(\rho)}$

Hence $OC = a \sin. \theta \frac{\sqrt{1+b(\rho)}}{\sqrt{1+b\rho}}$

and $OP = r \frac{\sqrt{1+b\rho - \frac{a^2}{r^2} \sin. \theta \left(1+b(\rho)\right)}}{\sqrt{1+b\rho}}$

Therefore $\dot{R} = \frac{\dot{O}C}{OP}$

$$= \frac{-\dot{\rho}\, a\, b \sin. \theta \sqrt{1+b(\rho)}}{2(1+b\rho)\, r \sqrt{1+b\rho - \frac{a^2}{r^2} \sin.^2 \theta \left(1+b(\rho)\right)}} \qquad (2)$$

This is Laplace's fundamental equation (3) vid. Méc. Cél. tom. 4, p. 244. b here corresponding to $\frac{4K}{n^2}$ in Laplace's formula.

2. The integral of this equation from $\varrho = (\varrho)$ to $\varrho = 0$ gives the atmospherical refraction required. It is obvious that to obtain the complete integral, it is necessary to know the relation between r and ϱ, or the law of diminution of the density of the atmosphere. This is at present unknown; but notwithstanding, we can approximate sufficiently to the value of R for all values of θ less than about 80°.

From the zenith to 74° zenith distance the result is the same whether we approximate to the integral, without knowing the relation of r and ϱ, or whether we assume any given relation, and reduce equation (2) to a convenient form for finding the integral.

Also by assuming two certain laws of variation of density we may obtain two integrals, one of which must give the refraction greater than the truth, and the other less. We find that as far as 80° 45′, * these refractions do not differ by one second, therefore a mean of the two must always give the refraction true within half a second so far from the zenith.

* The apparent zenith distance of the bright star, Capella, when below the pole, is in this latitude =80°45′, and having made many observations of this star S. P. I have taken that zenith distance as a limit.

Approximate integration of the Fluxional Equation.

3. Let Q represent the refraction that would take place if the surface of the earth were a plane, and the different *strata* of air parallel thereto, in which case the ratio of a to r would be the ratio of equality. Therefore equation (2) becomes $\dot{Q} = \frac{-\dot{\rho}\, b \sin.\theta \sqrt{1+b(\rho)}}{2(1+b\rho)\sqrt{1+b\rho-(1+b(\rho))\sin.^2\theta}}$ (3)

Hence $\dot{R} = \frac{\dot{Q}\, a\sqrt{1+b\rho-(1+b(\rho))\sin.^2\theta}}{r\sqrt{1+b\rho-(1+b(\rho))\frac{a^2}{r^2}\sin.^2\theta}}$

Let $\frac{a}{r} = 1-s$ (4)

Then $\dot{R} = \frac{\dot{Q}(1-s)}{\sqrt{1+\frac{(2s-s^2)(1+b(\rho))\sin.^2\theta}{1+b\rho-(1+b(\rho))\sin.^2\theta}}}$ (5)

or $\dot{R} = \dot{Q}(1-s)(1-s\tan.^2\theta) = \dot{Q} - \frac{\dot{Q}\, s}{\cos^2\theta}$ neglecting the second and higher powers of s, also ρ, (ρ) and their powers. It is obvious that for the part of the atmosphere which makes the refraction sensible, s must be very small.

By equat. (3)

$\dot{Q} = -\frac{1}{2} b\dot{\rho}\tan.\theta$ neglecting ρ, (ρ) and their powers.

Hence $R = Q + \int\frac{\dot{\rho}\, b\, s \tan.\theta}{2\cos.^2\theta}$ nearly. (6)

Now $\int\dot{\rho}\, s = \rho s - \int\rho\dot{s} = \rho s - \int\frac{\rho\dot{r}a}{r^2}$ (by equat. 4.)

Let p = the pressure of a column of superincumbent air of a given base, at the distance r from the centre. Then the pressure of a particle of air being measured by its magnitude, density and gravity, supposing the gravity at the surface represented by unity

$$-\dot{p} = \frac{\rho \dot{r} a^2}{r^2}$$

Hence $R = Q + \left(\varrho s + \frac{p}{a}\right) \frac{b \tan.\theta}{2 \cos.^2 \theta} +$ Constant.

when $R = o$, Q and $s = 0$ and $p = l\,(\varrho)$.

Therefore constant $= -\frac{b(\rho) l \tan.\theta}{2\, a \cos.^2 \theta} = -\frac{(m^2-1)\, l \tan.\theta}{2\, a \cos.^2 \theta}$

consequently the whole fluent from $\varrho = (\varrho)$ to $\varrho = 0$ is

$R = Q - \frac{(m^2-1)\, l \tan.\theta}{2\, a \cos.^2 \theta}$ or because m is nearly = unity

$$R = Q - \frac{(m-1)\, l \tan.\theta}{a \cos.^2 \theta}. \tag{7}$$

This expression as will be shewn farther on can be easily reduced to that of Laplace (Méc. cél. tom. 4. p. 268.) But it remains to shew how far from the zenith it can be used without inducing an error greater than a small fraction of a second.

4. The principal part Q of this expression is, it is evident, the deviation of a ray of light refracted at a given incidence θ from air of the density (ϱ) into a vacuum, and hence is entirely independent of the variation of density in the atmosphere. When m is known Q is known. The method of finding m will be considered hereafter.

The seconds in the latter part of the expression $=$ $\frac{(m-1)\, l \tan. \theta}{a \cos.^2 \theta \sin. 1''}$. To compute this quantity it is necessary to know m, l and a but not with much precision.

If we take $\theta = 80°$ and use, for the present, round numbers, taking $m = 1{,}0003$ and $\frac{l}{a} = \frac{5\ miles}{4000} = \frac{1}{800}$, $\frac{(m-1)\, l \tan. \theta}{a \cos.^2 \theta \sin. 1''} = 14''$ nearly. The terms which have been neglected, must obviously be much less. The limit may be thus computed.

Let the equations (3) and (5) of the last article be expanded, neglecting products of *three* dimensions of s, ϱ and (ϱ) and we shall obtain

$$\dot{R} = \dot{Q} + \frac{\dot{\rho}\, b \tan. \theta}{2 \cos.^2 \theta}\Big(s - b \varrho s - \tfrac{3}{2} s^2 \tan.^2 \theta + b s (\tan.^2 \theta + \tfrac{1}{2} \sec.^2 \theta)((\varrho) - \varrho)\Big)$$

Now of the terms that compose the factor of $\frac{\dot{\rho}\, b \tan. \theta}{2 \cos.^2 \theta}$, the first s has already been considered and found not to produce in integrating a quantity greater than a few seconds, as far as $\theta = 80°$; therefore after integration, the 2d and 4th on account of the smallness of $b\,(\varrho)$ and $b\,\varrho$ must be quite insensible; but the third $-\frac{3}{2} s^2 \tan.^2 \theta$, will produce a term $\int - \frac{3 \dot{\rho}\, b s^2 \tan.^3 \theta}{4 \cos.^2 \theta} =$
$- \frac{3 \rho\, b s^2 \tan.^3 \theta}{4 \cos.^2 \theta} + \int \frac{3 \rho\, b s \dot{s} \tan.^3 \theta}{2 \cos.^2 \theta}$.

The law of decrease of the density of the atmosphere is between that which a uniform temperature gives, and that of the density decreasing uniformly, as will be shewn further on. The true value of the above integral will therefore be

between the values deduced from an uniform temperature and an uniform density.

(1) For an uniform temperature. The density on this hypothesis is as the compressing force, and we have the well known equation

$$\varrho = (\varrho)\, c^{\left(\frac{a}{r}-1\right)\frac{a}{l}} \quad \text{where } c = 2{,}7128 \text{ \&c.}$$

$$\text{or } \varrho = (\varrho)\, c^{-\frac{as}{l}}$$

Hence $\int \frac{3\,\rho\, b\, s\, \dot{s}\ tan.^3\theta}{2\,cos.^2\theta} = \frac{3\,(\rho)\, b\, tan.^3\theta}{2\,cos.^2\theta} \int s\dot{s}\, c^{-\frac{as}{l}}$

$$\int s\dot{s}\, c^{-\frac{as}{l}} = -\frac{l}{a}\, s\, c^{-\frac{as}{l}} - \frac{l^2}{a^2}\, c^{-\frac{as}{l}} + \frac{l^2}{a^2} \text{ from } s = o$$

Therefore from $s = o$ to $s = i$ and from $\varrho = (\varrho)$ to $\varrho = o$

$\int \frac{3\,\varrho\, b\, s^2\, tan.^3\theta}{4\,cos.^2\theta} = \frac{3\,(\rho)\, b\, tan.^3\theta}{2\,cos.^2\theta} \cdot \frac{l^2}{a^2}$ having taken $c^{-\frac{a}{l}} = o$ on account of its extreme smallness, it being $= \frac{1}{(2{,}7128)^{800}}$

whence the term in question produces a quantity in seconds $=$ $\frac{3\,l^2\,(m-1)\,tan.^3\theta}{a^2\,cos.^2\theta\, sin.\, 1''}$

Taking $\theta = 80^\circ\ 45'$, $\frac{l}{a}$ and m as before

this quantity $= 2'',60$

Taking $\theta = 74^\circ$

It $= 0'',16$ a quantity not requiring notice.

(2) If the density of the air decrease uniformly, it will be proved that

$$s = \frac{(\rho) - \rho}{(\rho)} \times \frac{2l}{a} \text{ nearly}$$

$$\text{Hence} \int - \frac{3\,\dot{\rho}\, b\, s^2 \tan.^3 \theta}{4 \cos.^2 \theta} = \int - \frac{3\,\dot{\rho}\, b \tan.^3 \theta}{\cos.^2 \theta} \left(\frac{(\rho) - \rho)}{(\rho)}\right)^2 \frac{l^2}{a^2}$$

$= [\text{from } \rho = (\rho) \text{ to } \rho = o]\ \frac{b\,(\rho) \tan.^3 \theta}{\cos.^2 \theta} \times \frac{l^2}{a^2} =$ [in seconds] $\frac{2\,(m-1)\, l^2 \tan.^3 \theta}{a^2 \cos.^2 \theta \sin. 1''}$

Taking $\theta = 80^\circ\, 45'$ this quantity $= 1'',73$. Consequently the true value of $\int \frac{3\,\dot{\rho}\, b\, s^2 \tan.^3 80^\circ 45'}{4 \cos.^2 80^\circ 45'}$ is between $2'',60$ and $1'',73$ and therefore the mean cannot err quite half a second from the truth, and so the following formula may be considered as giving the refraction as far as 80° 45′ true to less than half a second, viz.

$$\text{Refraction} = Q - \frac{(m-1)\, l \tan. \theta}{a \cos.^2 \theta \sin. 1''} + \frac{5\,(m-1)\, l^2 \tan.^3 \theta}{2\, a^2 \cos.^2 \theta \sin. 1''}. \quad (7)$$

The third term is insensible when θ is less than 74° and the second and third insensible when θ is less than 40°

It is evident that the two first terms *must* be derived from assuming *any law of variation* of density, and then investigating the quantity of refraction as far as these terms. The following investigations in different hypotheses of density may be considered useful.

Hypothesis of uniform density.

5 Let CR be the radius of the uniform atmosphere, the height of which is l (vid. Fig.)

θ' = angle of incidence at the point R; $t = VRC$, then ref. $(R) = \theta' - t$, and $\frac{a}{a+l}\sin.\theta = \sin.t = \frac{\sin.\theta'}{m}$ (1)

Hence $a\,m\sin.\theta = (a+l)\sin.(t+R)$ (2)

but supposing the surface of the earth a plane

$$m\sin.\theta = \sin.(\theta+Q) \qquad (3)$$

Hence $\sin.(t+R) = \frac{\sin.(\theta+Q)}{1+\frac{l}{a}}$ (4)

making l, t and R to vary, in order to apply Taylor's Theorem.

By equat. (4)

$$(\dot{t}+\dot{R})\cos.(t+R) = \frac{-\dot{l}}{a\left(1+\frac{l}{a}\right)^2}\sin.(\theta+Q)$$

By equat. (1)

$$\dot{t}\cos.t = \frac{-\dot{l}}{a\left(1+\frac{l}{a}\right)}\sin.\theta$$

Hence computing $R+\dot{R}+$ &c. making $R = Q$, $t=\theta$ $\frac{l}{a}=o$ and then $\frac{\dot{l}}{a}=\frac{l}{a}$, we have by Taylor's Theorem

$$R = Q - \frac{l}{a}\left(\tan.(\theta+Q) - \tan.\theta\right) + \text{\&c.} \qquad (5)$$

But $\tan.(\theta+Q) = \tan.\theta + \frac{Q}{\cos.^2\theta} + $ &c.

Also making m and Q vary in equation (3)
We get by help of Taylor's Theorem

$\dot{Q} = (\dot{m}-1)$ tan. θ &c.

Hence substituting in equat. (5)

$R = Q - \frac{(m-1)\, l \tan.\theta}{a \cos.^2 \theta}$ as was found before in art. 3.

Hypothesis of density decreasing uniformly.

6. By the density decreasing uniformly is understood, that the density is as the distance from the highest part of the atmosphere. It is obvious that in this hypothesis, not taking into consideration the variation of gravity, the height of the atmosphere will be double of that of an uniform atmosphere of an uniform gravity. And it is also obvious that the effect of the variation of gravity can be but small. Lest however there should be any doubt on this head, it will be safer to investigate the height of the atmosphere on this hypothesis, gravity being supposed to vary.

Let this height $= l'$

the pressure at any height $z = p$

the pressure at the surface $= (p)$

a, l, ϱ &c. as before.

Then $\dot{p} = \frac{-\dot{z} \rho a^2}{(a+z)^2}$, the gravity at the surface being represented by unity.

On this hypothesis.

$$\varrho = (\varrho) \times \frac{l'-z}{l'} \qquad (1)$$

Therefore $\dot{p} = \frac{-\dot{z}(\rho)\, a^2\, (l'-z)}{(a+z)^2\, l'}$

and by integration,

$$p = \frac{(\rho)\, a^2}{a+z} + \frac{a^2\, (\rho)}{l'} \text{ h. log. } (a+z) + \frac{a^3(\rho)}{l'(a+z)} + \text{const.}$$

Hence this integral from $z = l'$ to $z = o$ gives

$$(p) = (\varrho)\left(a - \frac{a^2}{a+l'} + \frac{a^2}{l'} \text{ h. log. } \frac{a}{a+l'} + \frac{a^2}{l'} - \frac{a^3}{l'(a+l')}\right)$$

The right hand side of this equation being expanded according to the powers of $\frac{l'}{a}$ there results

$$(p) = (\varrho)\left(\frac{l'}{2} - \frac{l'^2}{3a} \text{ \&c.}\right)$$

but $(p) = (\varrho)\, l$

Hence is easily deduced $l' = 2l + \frac{8\, l^2}{3\, a}$ nearly

Having obtained l' we immediately deduce by equat. (1) the relation between ϱ and r on this hypothesis,

viz. $a - r + 2\, l + \frac{8\, l^2}{3\, a} = \frac{\rho}{(\rho)}\left(2\, l + \frac{8\, l^2}{a^2}\right)$

Whence $\frac{r}{a} = 1 + \frac{(\rho)-\rho}{(\rho)}\left(\frac{2l}{a} + \frac{8l^2}{a^2}\right)$ or regarding only one dimension of $\frac{l}{a}$, $\frac{a}{r} = 1 - \frac{(\rho)-\rho}{(\rho)} \times \frac{2l}{a}$ (2)

or $\frac{a}{r} = \left(\frac{1+b\rho}{1+b(\rho)}\right)^{\frac{2\, l}{b\,(\varrho)\, a}}$ b being introduced to form the factor $b\, \varrho$.

Let $1 + b\varrho = x$, $1 + b(\varrho) = (x)$ and $\frac{2l}{b(\rho)a} = f$

Then equat. (2) of art. 1 gives

$$\dot{R} = \frac{-\dot{x}\,x^{f} \sin.\theta\,(x)^{\frac{1}{2}}}{2\,(x)^{f}\,x\sqrt{x - \frac{(x)\,x^{2f}}{(x)^{2f}}\sin.^2\theta}} =$$

$$\frac{-\dot{x}\,x^{\frac{2f-3}{2}} \sin.\theta}{2\,(x)^{\frac{2f-1}{2}}\sqrt{1 - \left(\frac{x}{(x)}\right)^{2f-1}\sin.^2\theta}}$$

This by integration gives

$$R = -\frac{1}{2f-1}\left(\text{Circ. Arc. rad. 1 and sin.} = \left(\frac{x}{(x)}\right)^{\frac{2f-1}{2}}\sin.\theta\right) +$$

constant.

When $R = o$, $\varrho = (\varrho)$

Therefore constant $= \frac{1}{2f-1}\,\theta$.

Hence the integral from $\varrho = (\varrho)$ to $\varrho = o$ gives

$$R = \frac{1}{2f-1}\,\theta - \frac{1}{2f-1}\left(\text{Circ. Arc. rad. 1 and sin.} = \frac{\sin.\theta}{\left(1 + b(\rho)\right)^{f-\frac{1}{2}}}\right) \qquad (3)$$

or nearly

$$\frac{\sin.\theta}{\left(1 + b(\rho)\right)^{f-\frac{1}{2}}} = \sin.\left(\theta - (2f-1)\,R\right)$$

This is equivalent to Simpson's Rule, page 58, Math. Dissert.

By the well known analogy between the sum and diff. of the sines of two arcs and the tangents of the $\frac{1}{2}$ sum, and $\frac{1}{2}$ diff. the equat. (3) gives

$$\text{Tan.}\ \frac{2f-1}{2}\ \text{R} = \frac{1}{2}\left(\frac{2f-1}{2}\right) b\,(\varrho)\ \tan.\left(\theta - \frac{2f-1}{2}\ \text{R}\right) \quad (4)$$

$$\text{or R} = \frac{b}{2}\,(\varrho)\ \tan.\left(\theta - \left(\frac{2l}{(\rho)\,a\,b} - \frac{1}{2}\right)\text{R}\right) =$$

$$\frac{m^2-1}{2}\ \tan.\left(\theta - \left(\frac{l}{a(m-1)} - \frac{1}{2}\right)\text{R}\right)^{*} \quad (5)$$

From equation (5) we may obtain the same conclusions as in art. 3.

For if the surface of the earth were a plane, equation (5) would become

$$\text{Q} = (m-1)\ \tan.\left(\theta + \tfrac{1}{2}\,\text{Q}\right)\ \text{nearly}$$

Also because R and Q are very nearly equal at all zenith distances less than 80°. By equat. (4)

$$\text{R} = (m-1)\ \tan.\left(\theta + \tfrac{1}{2}\,\text{Q} - f\,\text{Q}\right).$$

From this equation it readily appears that

$$\text{R} = (m-1)\ \tan.\left(\theta + \tfrac{1}{2}\,\text{Q}\right) - \frac{\overline{m-1}\,f\,Q}{\cos.^2\left(\theta + \frac{1}{2}Q\right)}.$$

Therefore $\text{R} = \text{Q} - \frac{(m-1)\ l\ \tan.\ \theta}{a\ \cos.^2\theta}$ as before in art. 3.

* The formula used by Bradley is $R = k\ \tan.\ (\theta - n\,R)$. He determined n from the comparison of the horizontal refraction, and the refraction at a given altitude. This would be exact if the density of the atmosphere decreased uniformly. But k and thence n may be determined by direct experiments on the refractive force of air, and also by observations of circumpolar stars at zenith distances not greater than 80°. With these values of k and n the refractions at the horizon and low altitudes may be computed, and are not found to agree with observations, therefore the density of the atmosphere does not decrease uniformly.

7. *Remark.* This last conclusion might have been very easily deduced from equat. (6) art. 3; but the above investigation has been used for the sake of deriving the formulas of Simpson and Bradley.

By equat. (4) art. 3 $s = 1 - \frac{a}{r}$

Therefore. by equat. (2) art. 6, $s = \frac{(\rho) - \rho}{(\rho)} \times \frac{2l}{a}$

Hence by equat. (6) art. 3.

$$R = Q - \int \frac{\dot{\rho}\,(\rho) - \rho\dot{\rho}}{(\rho)} \times \frac{l}{a} \times \frac{b \tan. \theta}{\cos,^2 \theta} = [\text{from } \varrho = (\varrho) \text{ to } \varrho = o]$$

$$Q - \frac{b\,l\,(\rho) \tan. \theta}{2\,a \cos.^2 \theta} = Q - \frac{(m-1)\,l \tan \theta}{a \cos.^2 \theta}.$$

Hypothesis of an uniform temperature.

8. By the equat. (6) art. 3 we also derive the same conclusion on the hypothesis of an uniform temperature, in which case, as has been stated art 4.

$$\varrho = (\varrho)\, c^{\frac{-as}{l}} \text{ or } \dot{\rho} = -\frac{a\dot{s}}{l}\,(\varrho)\, c^{-\frac{as}{l}}$$

Hence by equation (6) art. 3.

$$R = Q - \int \frac{a\,s\,\dot{s}}{l}\,(\varrho)\, c^{-\frac{as}{l}}\, \frac{b \tan. \theta}{2 \cos.^2 \theta} = (\text{from } s = o \text{ to } s = 1)$$

$$Q - \frac{b\,(\rho)\,l \tan \theta}{2\,a \cos.^2 \theta} \text{ (vid. art. 4.)} = Q - \frac{(m-1)\,l \tan \theta}{a \cos.^2 \theta} \text{ as}$$

before.

Reduction of the formula for refraction to one convenient for computation.—Comparison with Laplace's formula.

9. From the equation which takes place, supposing the surface of the earth a plane.

Viz. $m \sin. \theta = \sin. (\theta + Q)$

We obtain, making $\dot{m}$ constant,

$$\dot{m} \sin. \theta = \dot{Q} \cos. (\theta + Q)$$

$$0 = \ddot{Q} \cos. (\theta + Q) - \dot{Q}^2 \sin. (\theta + Q)$$

Hence making $Q = 0$ and then $\dot{m} = m - 1$ we have by Taylor's theorem

$$Q = (m-1) \tan. \theta + \frac{(m-1)^2}{2} \tan.^3 \theta + \&c.$$

taking $m-1 = ,0003$ and $\theta = 80°.45'$

$$\frac{(m-1)^2 \tan.^3 \theta}{2 \sin. 1''} \quad 2'',1$$

the following terms are therefore insensible.

Hence substituting in equat. (7) art. 4.

We obtain for all values of θ less than about $80°.45'$

$$R = \frac{(m-1) \tan. \theta}{\sin. 1'} - \frac{(m-1)\, l \tan. \theta}{a \cos.^2 \theta \sin. 1''} + \frac{5\,(m-1)\, l^2 \tan.^3 \theta}{2\, a^2 \cos.^2 \theta \sin. 1''} + \frac{(m-1)^2 \tan.^3 \theta}{2 \sin. 1''} \qquad (1)$$

The two last terms are insensible except when θ is nearly $80°$.

10. The formula of Laplace (p. 268. tom. 4. Méc. célest.) in seconds of a degree $= \dfrac{\alpha \tan. \theta}{\sin. 1''} \left\{ 1 + \dfrac{\frac{1}{2} \alpha\,(2 \cos.^2 \theta + 1) - \frac{l}{a}}{\cos.^2 \theta} \right\}$

in which $\alpha = \frac{2K(\rho)}{1+\frac{4K}{n^2}(\rho)} = \frac{\frac{1}{2}b(\rho)}{1+b(\rho)}$

But $\frac{\frac{1}{2}b(\rho)}{1+b(\rho)} = \frac{m^2-1}{2m^2}$.

Therefore expanding $\frac{m^2-1}{2m^2}$ by the powers of $m-1$

$\alpha = (m-1) - \frac{1}{2}(m-1)^2$ &c.

substituting this value for α in Laplace's formula.

Ref. $= \frac{(m-1)\tan.\theta}{\sin.1''} - \frac{(m-1)\,l\tan.\theta}{a\cos.^2\theta\sin.1''} + \frac{(m-1)^2\tan.^3\theta}{2\sin.1''}$, the same as equation (1) article 9, excepting the term there introduced to make the formula applicable as far as $\theta = 80^\circ.45'$

Value of $\frac{m-1}{\sin.1''}$, and of $\frac{l}{a}$ — *Tables of Refraction.*

11. The refractive force of air being assumed proportional to its density, the value of m is variable, and its changes are known by the variations of the barometer and thermometer.

Let m' be the value of m when the height of the barometer $= 29{,}60$ inches, and the height of Farenheit's thermometer $= 50^\circ$. Let also b represent the height of the barometer, and t the height of the thermometer corresponding to m.

It appears by the results of the experiments of Dalton and Gay Lussac, that a column of air denoted by unity at the temperature of 32° of Farenheit becomes 1,375 at the

temperature of boiling water. In fact this agrees nearly with Mayer's conclusions made long before. It becomes therefore for t degrees of the thermometer $= 1 + ,002083\ (t - 32)$ It is not probable that the ratio of expansion is sensibly changed at different heights of the barometer within the limits of its usual variation. That is the ratio of the volumes at 32° and 212° is the same when the barometer is $28\frac{1}{2}$ inches as when $30\frac{1}{2}$ inches.

The increase of height in the barometer from the expansion of mercury by increase of temperature may be considered $\frac{1}{10000}$ for every degree of the thermometer.

Hence

$m - 1 : m' - 1 ::$ density of air bar. b and therm. t : density bar. 29,60 and therm. $50° :: b \times \left(1 - (t - 50) \times ,0001\right) \times 1,0375 : 29,60 \times \left(1 + ,002083\ (t - 32)\right)$

Therefore $m - 1 = (m' - 1) \times \frac{b}{29,60} \times \left(1 - (t - 50),0001\right) \times \frac{1,0375}{1 + ,002083\ (t - 32)}$.

The quantity $(m' - 1)$ may be deduced from the experiments of Biot and Arago (Mem. Inst. tom. 7.) who have most carefully repeated the experiments of Hawksby, or who rather by a different process, have accurately determined the refractive force of air. They have found when the height of barom. $= 0,76$ metre and centesimal therm. $= 0$ that is barom. 29,93 inches and F. thermometer $= 32°$ that

$m - 1 = ,0002946$.

Hence $m'-1 = \frac{,0002946 \times 29,60}{1,0018 \times 1,0375 \times 29,93} = ,0002803$

And $\frac{m'-1}{sin.\ 1''} = 57'',82.$

The height of an uniform atmosphere is not affected by the variation of the barometer, and therefore, if l' represent the height of an uniform atmosphere, the thermometer being 50°

$$l = l' \times \frac{1+,00208\ (t-32)}{1,0375}$$

a very accurate value of $\frac{l'}{a}$ is not required. If we take $l' = 5$ miles and the semidiameter of the earth $= 4000$ miles $\frac{l'}{a} = ,00125$. This in fact is sufficiently accurate.

But it will be more exact to take $l' = 5,095$ miles and the semidiameter of the earth $= 3979$ miles, and then $\frac{l'}{a} =$ $\frac{5,095}{3979} = ,00128.$

It is evident that the third and fourth terms of the value of the refraction in equat. (1) art. 9. cannot be sensibly affected by the variation of m, and therefore its mean value may be used as to these terms.

Hence substituting for $\frac{m-1}{sin.\ 1''}$ and $\frac{l}{a}$ in equat. (1) art. 9. we have

$$Refraction = \frac{1,0375}{1+,002083\ (t-32)} \times \left(1-,0001\ (t-50)\right) \frac{b}{29,60} \times 57'',82 \tan.\theta - \frac{b}{29,60} \times 0'',0739 \frac{tan.\ \theta}{cos.^2\ \theta} + 0'',000238 \frac{tan.^3\ \theta}{cos.^4\ \theta} + 0'',0080 \tan.^3\ \theta.$$

It is worthy of notice that the second term is independent of the thermometer, this circumstance enables us to put the three last terms into a very convenient table, the arguments of which are the zenith distance and height of the barometer.

12. The above expression for atmospheric refraction is entirely independent on astronomical observations.

The French tables are derived from observations of circumpolar stars. By these tables the refraction at $45° = 57'',57$ when the barometer shews 29,60 and Farenheit's thermometer 50° Hence by equat. (1) art. 9.

$$57'',57 = \frac{(m'-1)}{sin.\ 1''}\left(1-\frac{2\,l'}{a}\right) = \frac{m'-1}{sin.\ 1''}(0,99744).$$

Therefore $\frac{m'-1}{sin.\ 1''} = 57'',72$.

By 525 observations of circumpolar stars made by myself with the eight feet astronomical circle (vid. art. 14.) I deduce $\frac{m'-1}{sin.\ 1''} = 57'',56$.

Thus the value of $\frac{m'-1}{sin.\ 1''}$ by the French tables is between the values resulting from direct experiment and from my observations. I am inclined to give the preference to the result from direct experiment for reasons afterwards mentioned. But the difference between this result, and that from the French tables is so small that no inconvenience can occur in adopting the French tables. Thus, bar. 29,60 inches, and Farenheit's therm. 50°.

Zenith distance.	Refraction deduced from the experiment.	Refraction by the French Tables.
°	″	″
45	57,7	57,6
50	68,7	68,6
60	99,7	99,4
70	157,3	157,0
74	198,6	198,2

Therefore, as it is of considerable importance, particularly with a view of comparing observations made in different places, that the same refractions should be generally used, no objection, I apprehend, can be made to the general adoption as far as about 80° of the French refractions which are now so well known,

13. Perhaps the following tables deduced from the above formula, may be considered rather more convenient in many instances than the French tables; they will certainly furnish a useful check. The advantage they afford is derived from the facility with which the computation can be made by help of tables of logarithms and of logarithmic tangents to four or five places of figures, such as are in the "tables requisite to be used with the nautical ephemeris." By these the log. tangent of the zenith distance can be taken out at once, and the inconvenience of proportioning for the minutes of zenith distance avoided, which is greater than the new inconvenience occasioned by the second table. Hence the tables here given

may be considered more convenient for observations of the sun, moon, and planets.

In computing these tables 57″,72 was substituted in the above formula instead of 57″,82, and therefore the refraction deduced from these tables will agree with those deduced by the French tables.

TABLES FOR REFRACTION.

Table 1.

Far. Therm. °	Logarithms.	Far. Therm. °	Logarithms.	Far. Therm. °	Logarithms
10	0.3283	34	0.3048	58	0.2827
11	0.3273	35	0.3039	59	0.2818
12	0.3263	36	0.3030	60	0.2809
13	0.3253	37	0.3020	61	0.2800
14	0.3243	38	0.3011	62	0.2791
15	0 3233	39	0.3001	63	0.2782
16	0.3223	40	0.2992	64	0.2773
17	0.3213	41	0.2983	65	0.2764
18	0.3203	42	0.2974	66	0.2755
19	0.3193	43	0.2965	67	0.2746
20	0.3183	44	0.2956	68	0.2737
21	0.3173	45	0.2946	69	0.2728
22	0.3163	46	0.2937	70	0.2720
23	0.3154	47	0.2928	71	0.2711
24	0.3144	48	0.2919	72	0.2703
25	0.3134	49	0.2910	73	0.2694
26	0.3124	50	0.2900	74	0.2685
27	0.3114	51	0.2891	75	0.2677
28	0.3105	52	0.2881	76	0.2668
29	0.3095	53	0.2872	77	0 2660
30	0.3086	54	0.2863	78	0.2652
31	0.3076	55	0 2854	79	0.2644
32	0.3067	56	0.2845	80	0.2636
33	0.3058	57	0.2836	81	0.2627

Table 2. Barometer.

Z. D. °	28,50 "	29,00 "	29,50 "	30,00 "	30,50 "
80	10,5	10,7	10,9	11,1	11,4
79	8,1	8,3	8,5	8,7	8,9
78	6,3	6,4	6,6	6,7	6,9
77	5,1	5,2	5,3	5,4	5,6
76	4,1	4,2	4,3	4,4	4,5
75	3,4	3,4	3,5	3,6	3,7
74	3,0	3,0	3,1	3,1	3,2
73	2,5	2,5	2,6	2,6	2,6
72	2,1	2,1	2,2	2,2	2,2
71	1,8	1,8	1,9	1,9	1,9
70	1,5	1,5	1,5	1,6	1,6
69	1,3	1,3	1,3	1,4	1,4
68	1,2	1,2	1,2	1,2	1,2
67	1,0				1,0
66	0,9				0,9
65	0,8				0,8
64	0,7				0,7
63	0,6				0,6
62	0,6				0,6
61	0,5				0,5
60	0,5				0,5
58	0,4				0,4
56	0,3				0,3
54	0,3				0,3
52	0,2				0,2
50	0,2				0,2
45	0,2				0,2
40	0,1				0,1
30	0,0				0,0
0	0,0				0,0

Logarithm in Tab. 1. + log. barom. + log. tan. zenith dist. = log. approximate refraction.

Appr. ref.—Number Tab. 2. = refraction.

Example. Zenith dist. 71°. 26′, barom. 29,76 inches and therm. 43°.

Log. Tab. 1	- -	0.2965	Appr. ref. 175″,4
Log. barom.	- -	1.4736	Tab. 2. 2, 0
Log. tan. 71°.26	- -	0.4738	
			Ref. 173,4 =2′.53″,4
Log. approx. ref. 175″4	-	2.2439	

The Co-latitude of the Observatory of Trinity College, Dublin, deduced from Observations of Circumpolar Stars, by different Tables of Refraction.—Observed Refractions of Capella, below the Pole.

14. Comparisons of the Co-latitude as determined by stars near to, and remote from the pole, serve for a criterion of the accuracy of the tables of refraction used.

In the following table the co-latitude is determined by four different methods of computing the refraction.

1. In column A, by the formula $56'',9$ tan. $(\theta—3, 2 \text{ ref.}) \times \frac{bar.}{29,6} \times \frac{500}{450 + therm.}$.

2. In column B, by the formula $56'',9$ tan. $(\theta—3 \text{ ref.}) \times \frac{bar.}{29,6} \times \frac{400}{350 + therm.}$.

3. In column C, by the preceding tables, which give the same results as the French tables.

4. In column D, by the value of $\frac{m'-1}{sin. 1''} = 57'',82$ as deduced from experiment.

The second formula is Bradley's.

The first formula is what appeared to me by my observations in 1809, to give the refraction at low altitudes more exactly than Bradley's formula, and also to give the effects of the changes of temperature more exactly.

But both these formulæ must be considered empirical. We are entirely unacquainted with the law of variation of density at different heights, and therefore as has been shewn we cannot deduce from theory a formula of refraction that will serve much below 80°. It has been shewn indeed, art. 6. that if the density decrease uniformly, the refraction may be expressed by a similar formula, and that above 80° the refraction will not be sensibly changed by any law of variation of density; but then if $56'',9$ be the constant quantity, the co-efficient of refraction *must* be 4,14,* that is the mean ref. $= 56'',9$ tan. $(\theta - 4{,}14$ ref.) Therefore the two formula used in columns A and B are certainly inexact for all zenith distances less than about 80°. For greater zenith distances, the first formula will perhaps be found as exact as any other now known, at least as far as 87° 40'. But I do not attach much importance to it. I had deduced it before I was so well convinced as I am at present of the little value of observations near the horizon, and I may add of the impossibility of investigating an exact formula.

The mean of column C gives 36°. 36'. 46",54 for the co-latitude of the observatory or 53 23 13,46 for the latitude, which I conceive cannot possibly err $\frac{1}{4}$ of a second from the truth.

* For if $\frac{m-1}{\sin. 1''} = 56'',9$ $m-1 = ,0002758$, and therefore $\frac{l}{a(m-1)} - \frac{1}{2} = 4{,}14$ vid. art. 6. equat. (5).

The co-latitudes are each determined by a mean of the number of observations of each star above and below the pole as annexed.

Names of Circumpolar Stars	Obs. above Pole.	Obs. below Pole	Co-lat. A	Co-lat. B	Co-lat. C	Co-lat. D
Polaris	62	74	36°.36′.45″,65	36°.36′.45″,71	36°.36′.46″,19	36°.36′.46″,26
β Ursæ min.	20	18	46,18	46,42	46,77	46,85
β Cephei	10	10	45,43	45,64	46,37	46,46
α Ursæ maj.	10	8	46,91	47,19	47,33	47,44
α Cephei	10	9	45,56	45,71	46,42	46,53
β Ursæ maj.	21	21	45,49	45,95	46,62	46,75
ε Ursæ maj.	24	23	45,21	45,52	46,22	46,36
α Cassiopeæ	21	23	45,90	45,81	46,64	46,79
ζ Ursæ maj.	8	10	45,10	45,30	46,00	46,15
γ Ursæ maj.	18	21	45,81	46,18	46,85	46,90
γ Draconis	32	32	45,93	46,69	47,08	47,27
η Ursæ maj.	10	10	44,90	45,40	46,20	46,40
α Persei	10	10	44,53	44,35	46,29	46,51
Mean	256	269	36.36.45,58	36.36.45,84	36.36.46,54	

By 226 observations in 1808 and 1809 I had deduced
for column A 36°. 36′: 45″,65
B 45 ,85
C. 46 ,54

15. Let $c =$ the correction of 57″,82, that is, let $\frac{m'-1}{\sin. 1''} = 57'',82 + c$, then by comparing the co-latitudes in column D determined by Polaris, β Ursæ minoris and β Cephei with the same determined by the other ten stars, we have

$36^{\circ}.\ 36'.\ 46'',52 + ,82\ c = 36.\ 36.\ 46,71 + 1,56\ c$. The co-efficients of c are obtained from the tangents of the respective zenith distances.

This equation gives $c = -\frac{0''19}{,74} = -\ 0'',26$

and therefore $\frac{m'-1}{sin.\ 1''} = 57'',56$. By which, the mean refraction at $45^{\circ} = \frac{m'-1}{sin.\ 1''}\left(1 - \frac{2l'}{a}\right) = 57'',42$

Now from the number of observations used, it cannot be doubted that the above conclusion is free from the errors of observation. The only error by which it can reasonably be supposed affected, is that arising from errors of division.

It is difficult to state the limit of error from hence arising, but it will readily appear that much dependence cannot be had on a correction so small as that which I have deduced. For each star or each co-latitude, 12 points of the circle are used so that the quantity $36^{\circ}.\ 36'.\ 46'',52$, the mean of the results of the three first stars is affected by the mean error of 36 points of divisions of the circle. This mean error must certainly be very small. Yet it is not improbable that it may amount at least to $0'',15$.

The error of the quantity $36^{\circ}.\ 36'.\ 46'',71$ must be smaller, being only affected by the mean error of 120 points, yet it is not improbable it may amount to $0'',04$ and so the whole quantity $0'',19$, the numerator of the valuc of c, will be accounted for.

Thus it appears that observations of circumpolar stars are not adapted for obtaining extreme accuracy, and that the quantity of mean refraction at 45° so determined cannot reasonably be *depended* on to less than a quarter of a second.

The direct experiment for determining the refractive force of air may be made independently of the divisions of an instrument. The whole quantity of refraction is ascertained, instead of the differences of refractions as in circumpolar stars. There are also other sources of accuracy by which the result may bere ndered very exact.

For the above reasons, the determination $\frac{m'-1}{sin.\ 1''} = 57'',82$ or the mean refraction at 45″ (bar. 29, 60 and therm. 60) = 57,67 apoear to me more to be relied on.

16. In deducing the above value of $\frac{m'-1}{sin.\ 1''}$ from the observations of circumpolar stars, I only used such stars as were less than 80° from the zenith when below the pole.

It is well known to those conversant in observations made with good instruments that near the horizon an irregularity in refraction hitherto unexplained shews itself. This commencing even at less zenith distances than 80°, is at first very small, but increases to a very considerable irregularity as we approach the horizon.

The bright star Capella being within the limits of this irregularity has not been used for the co-latitude. A considerable number of observations of this star below the pole have however been made by me, which may serve for two purposes.

(1) To shew the effects of the abovementioned irregularity of refraction, by which it appears that at zenith distances not greater even than 80°, no use can be made of observations for the nicer purposes of astronomy.

(2) As it is reasonable to suppose this unexplained irregularity* will disappear from a mean of a great number of observations, this star, which is just at the limit where the quantity of refraction ceases to be independent of the variation of density, may also serve as a criterion of the exactness of the value of $\frac{m'-1}{sin.\ 1''}$, or of the quantity of mean refraction.

The refraction observed and the refraction computed by the formula in Art. 11. are placed by the side of each other, and also the correction of the computed refraction to give the observed refraction. This correction is often far beyond the limit of the error of observation, and is to be attributed to the abovementioned irregularity of refraction.

* The hypothesis upon which refractions are computed is that the different strata of air are concentrical with the earth's surface, circumstances may be easily imagined to affect this hypothesis, with respect to low stars.

Refractions of Capella below the Pole.

Time of Observations.	Bar.	Ther. int.	Comput. Refrac.	Observed Refrac.	Corr. comp. ref.	Time of Observation.	Bar.	Ther. int.	Comput. Refrac.	Observed Refrac.	Corr. comp. ref.
1808, July 28	29,50	63	5′ 30,3″	5′ 28,8″	− 1,5″	1811, Jan. 23	30,33	32	6′ 3,4″	6′ 5,2″	+ 1,8″
Aug. 11	29,51	61	31,9	29,1	− 2,8	27	29,40	27	5 56,3	5 59,8	+ 3,5
23	29,97	67	32,6	31,3	− 1,3	28	29,32	24½	57,5	55,3	− 2,2
24	29,98	66	33,4	33,9	+ 0,5	July 1	29,64	61½	30,7	31,6	+ 0,9
30	29,16	62½	26,8	26,2	− 0,6	3	29,49	54½	36,5	42,9	+ 6,4
Nov. 23	29,84	42	49,7	42,4	− 7,3	6	29,78	61½	34,4	43,7	+ 9,3
Dec 4	29,77	44	47,4	43,5	− 3,9	9	29,81	64½	32,5	35,8	+ 3,3
21	29,30	31	52,1	47,8	− 4,3	14	29,42	58½	32,4	32,6	+ 0,2
1809, Jan 20	29,31	30	52,7	48,3	− 4,4	16	29,46	57½	34,6	36,2	+ 2,6
22	29,33	27	55,6	48,5	− 7,1	17	29,46	58	33,3	34,7	+ 1,4
May 29	29,50	54	36,7	43,0	+ 6,3	20	29,80	63½	33.2	35,3	+ 2,1
June 14	29,70	54	38,9	41,6	+ 2,7	21	29,73	64	32,1	29,4	− 2,7
15	29,72	55	38,4	38,8	+ 0,4	22	29,78	61	34,8	36,6	+ 1,8
17	29,61	56	36,5	38,6	+ 2,1	23	29,83	62	34,5	40,1	+ 5,6
July 8	29,90	63	34,6	39,1	+ 4,5	26	30,00	65½	34,1	37,5	+ 3·4
10	29,97	63	35,5	36,7	+ 1,2	Dec. 9	28,67	40½	37,2	34,0	− 3,2
15	29,88	62½	34.7	39,3	+ 4,6	9	28,87	38	41,4	37,0	− 4,4
17	29,80	55½	38,9	35,6	− 3,3	13	29.75	40	50,3	47,1	− 3,2
18	29,89	57½	38,4	41,0	+ 2,6	18	29,15	45½	39,0	34,1	− 4,9
19	29,92	60	37,1	38,3	+ 1,2	29	29,84	30½	58,8	54,2	− 4,6
23	29,71	60	34,7	35,5	+ 0,8	1812, Jan. 4	29,20	29½	51,9	43,7	− 8,2
Aug. 22	29,19	53	33,9	29,8	− 4,1	14	29,42	37	48,4	47,2	− 1.2
24	29,16	55	32,1	30,5	− 1,6	20	29,69	37	51.9	50,1	− 1,8
1810, Jan. 20	29,83	58½	37,2	40,0	+ 2,8	Oct. 28	29,33	46½	40,1	38,1	− 2,0
22	30,12	62	37,2	43,6	+ 6,4	Dec. 9	29,62	36	50,0	46,7	− 3,3
23	30,02	62½	36,4	42,0	+ 5,6	21	29,48	35	50,7	50,6	− 0,1
25	29,98	57	39,9	42,7	+ 2,8	31	29,57	40	47.9	47,8	− 2,1
July 1	29,58	58	34,6	39,4	+ 4,8	1813, Jan. 4	29,59	42	46,5	39,4	− 7,1
8	29,50	58	33,8	36,7	+ 2,9	11	29,36	34	50,0	42,7	− 7,3
24	29,73	59	35,7	29,9	− 5,8	18	29,88	36	54,7	52,0	− 2,7
27	29,24	58	30,4	25,2	− 5,2	19	30,02	35	57.2	57,3	+ 0,1
Aug. 14	29,29	58	31,4	27,3	− 4,1	25	30,15	29	6 3,5	6 3,6	+ 0,1
1811, Jan. 20	29,66	37	51,5	42,9	− 8,6						

The preceding 65 observations give the mean correction $= - 0'',49$. This would give $\frac{m'-1}{sin.\ 1''} = 57'',74$ and the ref. at $45° = 57'',58$ very nearly the same as the French tables, but this exactness cannot be depended on, even if we supposed the irregularity of refraction to disappear in the mean, because the zenith distances of Capella above and below the pole may be affected by errors of division. If we suppose the co-latitude exact, and take the error of the mean of the six microscopes in each position of Capella $= 0'',5$ and also take the error of refractiou arising from using the mean between uniform temperature and uniform density $= 0'',25$. The above correction may become $= -(0''49 + 1,00 + 0'',25)$
$= - 1'',74$
or it may become $+ 0,76$.

The first will make the ref. at $45° = 57'',37$
the second - - $= 57\ ,79$
These are probably two limits.

Limits of Refraction.—Observed Refractions of α Lyræ below the Pole.

17. It has been stated in art. 4. that the quantity of atmospherical refraction is less than would result from an uniform temperature in the atmosphere and greater than what would result from a density decreasing uniformly.

(1) The former readily appears from the equation

$$R = \frac{\dot{O}C}{OP} \text{ art. 1.}$$

For since the temperature decreases as we ascend, it follows that the *same* density takes place at a distance from the surface greater than in the case of an uniform temperature. Now the only variable quantity in OC is ϱ, therefore OC remaining the same, OP is increased, and consequently $\dot{R}$ diminished, therefore refraction or $\int \dot{R}$ is greater in the case of uniform temperature than in the actual state of the atmosphere.

(2) By the annexed observed refractions of α Lyræ, below the pole it will appear that the actual refraction is greater than would take place, did the density of the air decrease uniformly.

The mean of these 42 observations of α Lyræ below the pole gives the refraction at the zenith distance $87° \; 42' \; 10'' =$ $17' \; 20'',5$, the mean of the heights of the barom. $=$ 29,50, and the mean of the heights of the therm. $= 35°,0$.

These heights of the barom. and therm. give, (vid. art. 11.) $\frac{m-1}{\sin. 1''} = 59'',50$ and $\frac{l}{a(m-1)} - \frac{1}{2} = 3,803$. Hence if the density of the air decreases uniformly,

At $87°. 42'. 10''$, refraction $= 59'',5$ tan. $(87°. 42'. 10'' - 3{,}803\, r)$* $= 16'. 51'',0$.

This refraction is less by $35'',5$ than the mean of the observed refractions. Hence we may safely conclude that the actual quantity of refraction is between the results from an uniform temperature and from a density decreasing uniformly.

Laplace has shewn the same from the horizontal refractions computed on each hypothesis, and compared with the observed horizontal refraction. But it does not appear that the mean observed horizontal refraction has hitherto been ascertained with much accuracy.

Laplace has also in the case of uniform temperature integrated the fluxional equation for refraction, in which he

* This form or $r = k$ tan. $(\theta - n r)$ may be readily computed by help of an auxiliary angle y.

$$\log.\ \tan.\ y = \log.\ \tan.\ \theta + \log.\left(\frac{2k}{1 + nk\,\sin.\ 1''}\right) + \tfrac{1}{2}\log.\ \frac{n\,\sin.\ 1''}{k}$$

then $\log.\ r = \frac{1}{2}\log.\ \frac{k}{n\,\sin.\ 1''} + \log.\ \tan.\ \frac{1}{2}y$

For k tan. $(\theta - n r) = \dfrac{k\,\tan.\ \theta - k\,\tan.\ nr}{1 + \tan.\ \theta\,\tan.\ nr}$

Hence $\dfrac{(1 + nk\,\sin.\ 1'')\, r}{k\left(1 - \frac{nr^2}{k}\,\sin.\ 1''\right)} = \tan.\ \theta$

let $\tan.\ \frac{1}{2}y = r\sqrt{\frac{n}{k}\,\sin.\ 1''}$

then $\dfrac{1 + nk\,\sin.\ 1''}{2k}\sqrt{\dfrac{a}{k\,\text{siu}.\ 1''}}\ \tan.\ y = \tan.\ \theta$

Whence log. tan. $y =$ tan. θ + &c.

&c. &c.

has exhibited a striking specimen of his great mathematical skill (vid. Méc. cél. tom. 4. p. 246—253)

His series is sufficiently convenient for computing the horizontal refraction, but in deducing from it the refraction at 87°42′10″ zenith distance, a good deal of calculation is necessary. I deduce the value of α=,0002882 for the heights of the barometer and therm. abovementioned, and then the six first terms of the series (Méc. cél. tom. 4. p. 251)= 817″ + 171″,4 + 50″,2 + 17″,4 + 6″,4 + 2″,7 + &c.

The sum of this series must be nearly = 1067″.

Therefore we have at zen. dist. 87°42′10″, barom. 29,50 and therm. 35°.

Refraction,	density decreasing uniformly	= 16.′51″,0
	by observation -	= 17. 26, 5
	uniform temperature -	= 17. 47, 0

Hence as far as this zenith distance the refraction differs only a few seconds from the mean resulting from the two hypotheses. The difference is far less than what may arise from the irregularity of refraction.

At the same zenith distance, and same heights of the barom. and therm.

By the French tables ref.	= 17′.21″,0
By Bradley's formula	= 17 48, 2
By what I considered an improvement of Bradley's formula vid. art. 14	= 17 25, 3

Refractions of α Lyræ below the Pole.

Time of Observation	Barom.	Ther. int.	Ther. ex.	Zenith distance observed	Ref. observed	Corr. French Tables
				° ′ ″	′ ″	″
1809, Jan. 22	29,25	25		87 42 1,6	17 57,4	+ 23,7
Feb. 18	30,01	43½		42 40,7	17 24,8	+ 3,4
20	29,78	43½		42 41,6	17 24,2	+ 10,7
Mar. 5	30,09	42½		42 33,0	17 34,7	+ 8,7
12	30,05	44		42 22,1	17 46,2	+ 26,0
1810, Feb. 13	28,94	34	30	42 57,0	17 3,1	— 3,5
19	30,02	32	29½	42 5,9	17 55,6	+ 10,2
Mar. 17	29,62	36	33	42 31,0	17 33,4	+ 9,4
1811, Jan. 18	29,90	33½	32	42 12,2	17 38,1	— 0,2
23	30,27	35	32½	41 55,1	17 56,6	+ 9,1
28	29,35	27½	21½	41 58,5	17 54,6	+ 22,7
Feb. 3	29,44	31½	30	42 34,3	17 20,4	— 7,7
7	29,24	39	38	42 52,5	17 3,2	— 2,8
8	29,28	39	35	42 51,2	17 4,7	— 2,3
12	29,03	38	34	42 56,4	16 58,4	— 2,6
13	28,91	35	33	43 3,3	16 53,7	— 10,2
Dec. 28	29,39	30½	25½	42 3,0	17 38,7	+ 12,2
1812. Jan. 2	29,07	31½	30	42 22,0	17 21,2	+ 6,9
3	28,95	29½	26½	42 34,0	17 9,5	— 5,8
4	29,11	27½	23½	41 56,2	17 47,6	+ 24,6
7	29,93	32	31	42 2,1	17 42,6	+ 0,6
21	29,64	34	28½	42 1,2	17 47,9	+ 20,7
30	29,18	39	35	42 36,4	17 19,2	+ 17,2
Feb. 7	29,42	38	33	42 27,2	17 26,4	+ 13,9
Dec. 22	29,66	33	26½	41 48,0	17 50,7	+ 21,1
1813, Jan. 1	29,64	36	31	42 9,1	17 32,7	+ 9,4
3	29,90	42½	40	42 23,0	17 19,5	+ 0,7
11	29,52	36	31½	42 11,8	17 33,2	+ 14,2
19	30,04	36	32	41 58,2	17 49,2	+ 12,6
26	30,16	33	28	41 46,2	18 3,2	+ 16,1
Feb. 6	29,40	39	38	42 4[illegible],8	17 5,6	— 5,1
15	28,50	40	38	43 24.8	16 29,6	— 10,0
18	29,26	39	37½	43 0,0	16 55,0	— 12,0

Refractions of α Lyræ below the Pole.

Time of observation	Barom.	Ther. int.	Ther. ex.	Zenith distance observed	Ref. observed	Corr. French Tables
				° ′ ″	′ ″	″
1813, Feb. 22	29,24	42	36½	87 42 52,3	17 3,3	+ 4,0
Dec. 26	30,19	35½	31½	41 55,8	17 43,6	+ 0,1
27	30,01	36½	34	42 21,2	17 18,5	— 17,3
31	29,88	35½	33½	42 1,0	17 40,0	+ 7,5
1814, Jan. 1	29,69	35	32½	42 21,2	17 20,1	— 8,4
4	29,11	26½	23	41 59,6	17 42,7	+ 17,4
22	29,88	21	17	41 25,7	18 22,2	+ 18,2
26	28,95	33	32½	42 56,2	16 52,8	— 16,5
27	28,78	32½	30½	42 49,8	16 59,4	— 4,2
29	28,63	31½	29	42 51,5	16 58,4	— 2,1
Feb. 13	29,67	41½	39	42 47,1	17 6,3	— 8,4

To the preceding observed refractions of α Lyræ *S.P.* are annexed the corrections to be applied to the refractions computed by the French tables to give the observed refractions. These corrections sufficiently point out the irregularities of refraction at low altitudes.

The French tables from 74° zenith distance to the horizon may be considered less empirical than any other, since they are deduced from a formula of Laplace assumed so, that, partaking both of the arithmetical and geometrical progressions of variation of density, it gives the diminution of heat observed in ascending in the atmosphere. Gay Lussac having ascended in a balloon to a considerable height found the diminution of temperature nearly as resulted from Laplace's formula.

But from the circumstances of the case there seems to be no reason to expect any exact and convenient method of determining the quantity of refraction for low altitudes.

It is not likely the irregularities will be ever submitted to any law, and investigations respecting formulæ for refractions for zenith distances greater than about 80° may be considered more curious than useful. For less zenith distances, the French tables, as it has been a principal object of this paper to shew, seem as accurate as can be desired.